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THE ZERO–TWO LAW IN BANACH LATTICE ALGEBRAS

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ABSTRACT
Let A be a unital Banach lattice algebra and let $a \in A^+$ satisfy $\| a \| \leqq 1$. Then either $\| a^{n+1} - a^n \| = 2$ for all $n \geqq 0$ or else $\| a^{n+1} - a^n \| \to 0$ as $n \to \infty$. Cyclicity of the peripheral spectrum of a is also established.

1. Introduction

Since the inception of the zero–two law for positive contractions by R. Zaharopol [9] in certain L^p-spaces, a number of authors contributed to the extension of the result to arbitrary Banach lattices (see R. Wittmann [7], Y. Katznelson and L. Tzafriri [2]). The strongest results known thus far are due to H. H. Schaefer [4] and A. R. Schep [6]. Schaefer proved that in an arbitrary Banach lattice one has for a positive linear contraction T that either $\| T^{n+1} - T^n \|_r = 2$ or else $\| T^{n+1} - T^n \| \to 0$ as $n \to \infty$, and Schep showed that in a Dedekind complete Banach lattice one has for the second alternative the stronger result that $\| T^{n+1} - T^n \|_r \to 0$ as $n \to \infty$. We show that Schep's result holds in an arbitrary Banach lattice algebra, i.e., that if $a \in A^+$, A a unital Banach lattice algebra, and if $\| a \| \leqq 1$ then either $\| a^{n+1} - a^n \| = 2$, or else $\| a^{n+1} - a^n \| \to 0$ as $n \to \infty$. Our proof will not use Schaefer's results on the cyclicity of the peripheral spectrum of a positive operator and we consider it part of the contribution of this paper to the theory that, as far as these results go, our proofs are self-contained. In the first section we prove the results which are needed concerning the peripheral spectrum and in the second section we prove our main result.

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2. The peripheral spectrum of positive elements

We refer the reader to [8] and to [3] for notation and terminology concerning Banach lattices and lattice algebras. For the theory of Banach algebras the reader may consult [1]. We denote the spectrum of an element a in a Banach algebra A by $\sigma(a)$ (or $\sigma(a, A)$ if necessary). The peripheral spectrum $\sigma_r(a)$ of a is defined as the set

$$\sigma_r(a) := \{\lambda \in \sigma(a) : |\lambda| = r(a)\},$$

where $r(a)$ denotes the spectral radius of a. We also denote by $C(a)$ the commutative closed subalgebra of A generated by the unit 1 of A and a, and note that it easily follows from [1], I.5.12 that $\sigma_r(a, A) = \sigma_r(a, C(a))$.

A lattice algebra A with the property that $x \wedge y = 0$ implies that $ux \wedge y = xu \wedge y = 0$ for all $u \geqq 0$ is called an *f-algebra*. It is well known that if E is a Banach lattice and if $u \geqq 0$ then the principal ideal E_u generated by u in E is an f-algebra with unit u. The proof of the next result may be of some interest.

2.1. PROPOSITION. *Let E be a Banach lattice, let $\mu \in E + iE$ and let B denote the f-algebra $E_{|\mu|} + iE_{|\mu|}$. If $0 \leqq \Phi \in B'$ is such that $\Phi(|\mu|) = 1 = |\Phi(\mu)|$ then we have for every $\sigma \in B$ that*

$$\Phi(\mu\sigma) = \Phi(\mu)\Phi(\sigma).$$

Moreover, if Φ is strictly positive, then $\mu = \exp(i\alpha)|\mu|$ for some α.

PROOF. Let $\Phi(\mu) = \exp(i\alpha)$, $\mu_1 = \exp(-i\alpha)\mu = v + iw$, with $v, w \in B$. Then $\Phi(\mu_1) = 1$ and $\Phi(|\mu_1|) = \Phi(|\mu|) = 1$ with the result that $\Phi(|\mu_1| - \mu_1) = 0$ and hence $\Phi(|\mu_1| - v) = 0$ and $\Phi(w) = 0$. We prove that $\Phi(|w|) = 0$. Indeed, we have

$$|\mu_1| = \sup\{|v|\cos\beta + |w|\sin\beta : 0 \leqq \beta \leqq \pi/2\}$$

and hence for every $0 \leqq \beta \leqq \pi/2$, $|v|\cos\beta + |w|\sin\beta \leqq |\mu_1|$. This implies that

$$\sin\beta\Phi(|w|) \leqq \Phi(|\mu_1| - |v|\cos\beta) = (1 - \cos\beta)\Phi(|\mu_1|)$$

and since $\tan(\beta/2) \to 0$ whenever $\beta \to 0$, we obtain $\Phi(|w|) = 0$. Hence, supposing that $|\sigma| \leqq k|\mu|$, we get

$$0 \leqq |\Phi(\sigma w)| \leqq \Phi(|\sigma w|) \leqq \Phi(k|\mu|\,|w|) = k\Phi(|w|) = 0$$

and

$$|\Phi(\sigma|\mu_1| - \sigma v)| \leqq \Phi(|\sigma|(|\mu_1| - v)) \leqq k\Phi(|\mu_1| - v) = 0.$$

Consequently,

$$\Phi(\sigma\mu_1) = \Phi(\sigma v) + i\Phi(\sigma w) = \Phi(\sigma\,|\mu_1|) = \Phi(\sigma),$$

and finally

$$\Phi(\sigma\mu) = \exp(i\alpha)\Phi(\sigma\mu_1) = \Phi(\mu)\Phi(\sigma).$$

Assume now that Φ is strictly positive. Then it follows from $\Phi(|\mu_1| - v) = 0$ and $\Phi(|w|) = 0$ that $\exp(-i\alpha)\mu = v = |\mu_1| = |\mu|$ which is the desired result. ■

The following general result follows.

2.2. Proposition. *Let A be a unital Banach lattice algebra and let $\lambda \in \sigma_r(z)$, $z \in A^+$, $r(z) = 1$. If there exists a continuous linear functional $\mu \in A'$ satisfying $\mu(z^n) = \lambda^n = \mu(z)^n$ and $|\mu|(z^n) = |\mu|(z)^n$ for $n = 0, 1, \ldots$, then $\lambda^k \in \sigma_r(z)$ for all $k \in \mathbf{Z}$, i.e., $\sigma_r(z)$ is cyclic.*

Proof. Our assumptions on μ implies that both μ and $|\mu|$ are multiplicative linear functionals on the commutative closed algebra $C(z)$. It follows that $|\mu|(z) \in \sigma(z, C(z))$ and since the spectral radius of z remains unchanged in the algebra $C(z)$, we have $|\mu(z)| = 1 \geqq |\mu|(z)$ and so $|\mu(z)| = 1 = |\mu|(z)$. For every σ in the f-algebra $(A')_{|\mu|}$ we therefore get by the preceding proposition that $(\sigma\mu)(z) = \sigma(z)\mu(z)$. In particular, since $\{v : |v| = |\mu|\}$ is a multiplicative group in the f-algebra, $\mu^k(z) = \mu(z)^k$ for all $k \in \mathbf{Z}$. We now show that μ^k is multiplicative on $C(z)$. Let $n \in \mathbf{N}$. Then $\mu(z^n) = \mu(z)^n$ and so $|\mu(z^n)| = |\mu(z)|^n = 1$ and also $|\mu|(z^n) = |\mu|(z)^n = 1$. Since $z^n \geqq 0$, it follows from 2.1 that z^n is multiplicative in the sense described in 2.1. In particular, $\mu^k(z^n) = \mu(z^n)^k$ from which we derive that

$$\mu^k(z^n) = \mu(z)^{kn} = (\mu(z)^k)^n = (\mu^k(z))^n.$$

In the same manner, since $|\mu|(1) = 1 = |\mu(1)|$ and $1 \geqq 0$ we also get $\mu^k(1) = 1$. It follows that μ^k is multiplicative on $C(z)$. Hence, $\mu^k(z) \in \sigma(z)$. But, since we have $\mu^k(z) = \mu(z)^k$, it follows that $\lambda^k \in \sigma(z)$, and consequently, $\lambda^k \in \sigma_r(z)$ for all $k \in \mathbf{Z}$. ■

2.3. Corollary (E. Scheffold [5]). *Let A be a commutative unital Banach lattice algebra and suppose that for every multiplicative linear functional μ on A the functional $|\mu|$ is also multiplicative. Then $\sigma_r(z)$ is cyclic for every $z \in A^+$.*

Following standard terminology from operator theory, we call an element a in a Banach algebra A *normaloid* if $r(a) = \| a \|$.

2.4. THEOREM. *Let A be a unital Banach lattice algebra and let $z \in A^+$ be normaloid. Then z has a cyclic peripheral spectrum.*

PROOF. Let $z \in A^+$ and suppose without loss of generality that $r(z) = 1$. If $\lambda \in \sigma_r(z)$ then $\lambda - z$ is a topological divisor of zero, i.e., there exists a sequence $(a_n) \subset A$, $\| a_n \| = 1$ such that $\| (\lambda - z)a_n \| \to 0$ as $n \to \infty$. Choose $\phi_n \in A'$ such that $\phi_n(a_n) = 1$, $\| \phi_n \| = 1$ and define $\mu_n \in A'$ by $\mu_n(b) := \phi_n(ba_n)$ for each $b \in A$ and for all $n \in \mathbf{N}$. Then $\| \mu_n \| \leqq 1$ for all $n \in \mathbf{N}$. Let $\mu \in A'$ be a $\sigma(A', A)$-clusterpoint of the sequence (μ_n). We claim that μ satisfies the conditions of Proposition 2.2. Clearly, $\mu(1) = 1$. Furthermore, from the inequality

$$
\begin{aligned}
&|\mu(z^n) - \lambda\mu(z^{n-1})| \\
&\leqq |\mu(z^n) - \phi_k(z^n a_k)| + |\phi_k(z^n a_k) - \lambda\phi_k(z^{n-1}a_k)| \\
&\quad + |\lambda\phi_k(z^{n-1}a_k) - \lambda\mu(z^{n-1})| \\
&\leqq |\mu(z^n) - \mu_k(z^n)| + \| \phi_k \| \, \| z^{n-1} \| \, \| (\lambda - z)a_k \| \\
&\quad + |\lambda\mu_k(z^{n-1}) - \lambda\mu(z^{n-1})|
\end{aligned}
$$

we derive by induction that $\mu(z^n) = \lambda^n$ for $n = 0, 1, \ldots$. Finally,

$$1 = |\mu(z^n)| \leqq |\mu|(z^n) \leqq \| \mu \| \, \| z^n \| \leqq 1.$$

This completes the proof. ∎

REMARK. Proposition 2.2 can be used to prove the cyclicity of both the spectrum and the order spectrum of a positive operator on a Banach lattice if the operator satisfies the growth condition (G) of [3]. One can, in the presence of condition (G), show that a functional with the properties required in 2.2 exists. Note also that if a positive operator is normaloid, then it does satisfy condition (G).

2.5. COROLLARY. *If A is a unital Banach lattice algebra and if $z \in A^+$ is a contraction with $r(z) = 1$ then $\sigma_r(z)$ is cyclic.*

3. The zero–two law

Our first lemma contains the idea which lies at the heart of the zero-two law.

3.1. LEMMA. *Let A be a complex unital Banach lattice algebra and let $z \in A^+$ satisfy $\| z \| \leqq 1$. Let λ be an element of the unit circle in $\mathbf{C}$ such that the numbers $\lambda^{n+k_1}, \ldots, \lambda^{n+k_m}$ are distinct. If there exists a linear functional $\mu \in A'$, $\| \mu \| \leqq 1$ satisfying $\mu(z^n) = \lambda^n$ and $|\mu|(z^n) = 1$ for all $n \in \mathbf{N}$, and if $\varepsilon_j = \pm 1$, then*

$$\| \varepsilon_1 z^{n+k_1} + \cdots + \varepsilon_m z^{n+k_m} \| = m$$

for all $n \in \mathbf{N}$.

PROOF. Let B denote the band generated by $|\mu|$ in A' and let U_j denote the canonical image of z^{n+k_j} in the order continuous dual $B_{00}^{\sim}$ of B for $j = 1, \ldots, m$. Denote by C_j the carrier band of U_j in B and by P_j the band projection of B onto C_j for $1 \leqq j \leqq m$. From our hypothesis we get $|U_j(\mu)| = 1 = U_j(|\mu|)$ and it follows from 1.1 that $P_j\mu = \exp(i\alpha_j)P_j|\mu|$. But, since

$$P_j\mu(z^{n+k_j}) = \mu(z^{n+k_j}) = \lambda^{n+k_j} = \exp(i(n+k_j)\theta),$$

with $\theta \neq 0$, we may put $\alpha_j = (n+k_j)\theta$. We now claim that $U_p \wedge U_q = 0$ for $0 \leqq p, q \leqq m$. If not, $C_{pq} := C_p \cap C_q \neq \{0\}$ and therefore, if P_{pq} denotes the projection of B onto the band C_{pq}, $P_{pq}|\mu| \neq 0$ since $|\mu|$ is a weak order unit in B. If $x \in A$ is such that $P_{pq}|\mu|(x) > 0$, we get

$$\begin{aligned} P_{pq}\mu(x) = P_{pq}P_p\mu(x) &= \exp(i(n+k_p)\theta)P_{pq}P_p|\mu|(x) \\ &= \exp(i(n+k_p)\theta)P_{pq}|\mu|(x) \end{aligned}$$

and also in the same manner, $P_{pq}\mu(x) = \exp(i(n+k_q)\theta)P_{pq}|\mu|(x)$. This implies that $\lambda^{n+k_p} = \lambda^{n+k_q}$, contradicting our hypothesis. Finally, let

$$\nu := \sum_{j=1}^{m} \exp(-i\alpha_j)P_j\mu.$$

Then $|\nu| \leqq |\mu|$ and

$$\nu\left(\left|\sum_{j=1}^{m} \varepsilon_j z^{n+k_j}\right|\right) = \sum_{j=1}^{m} U_j(\nu) = \sum_{j=1}^{m} \exp(-i\alpha_j)P_j\mu(z^{n+k_j}) = \sum_{j=1}^{m} |\mu|(z^{n+k_j}) = m.$$

Hence, since $\| \nu \| \leqq 1$, and since z is a contraction,

$$\left\| \sum_{j=1}^{m} \varepsilon_j z^{n+k_j} \right\| = m. \qquad \blacksquare$$

3.2. THEOREM. *Let A be a complex unital Banach lattice algebra and let $z \in A^+$, $\| z \| \leqq 1$. Then either $\| z^n - z^{n+1} \| = 2$ for all $n \in \mathbf{N}$, or else $\| z^n - z^{n+1} \| \to 0$ as $n \to \infty$.*

PROOF. Suppose that $\| z^n - z^{n+1} \| \not\to 0$. Using the left regular representation $a \mapsto \lambda_a$ of A into $L(A)$, defined by $\lambda_a b := ab$ we have that λ_z is a contraction satisfying $\| \lambda_z^n - \lambda_z^{n+1} \| \not\to 0$. From a result of Katznelson and Tzafriri [2] it follows that $\sigma(\lambda_z)$ intersects the unit circle in at least one point $\lambda \neq 1$. Since $\sigma(\lambda_z) = \sigma(z)$ ([1]) and since the spectral radius $r(z) \leqq 1$ we have $\lambda \in \sigma_r(z)$, and $\lambda^n \neq \lambda^{n+1}$ for all $n \in \mathbf{N}$. Exactly as in the proof of 2.4 we find a functional μ which satisfies the conditions of Lemma 3.1 and our proof is complete. ■

The next two generalizations are due to A. R. Schep in the operator case. We refer to [6] for the simple proofs.

3.3. COROLLARY. *Let A be a complex unital Banach lattice algebra and z a normaloid element with $r(z) = 1$ satisfying $z^n \wedge z^{n+1} = 0$ for infinitely many n. Then $\| z^n - z^{n+1} \| = 2$ for all $n \in \mathbf{N}$ and $\sigma_r(z)$ consists of a finite union of non-trivial groups of roots of unity or equals $\{z : |z| = 1\}$.*

3.4. PROPOSITION. *Let A be a complex unital Banach lattice algebra and let $z \in A^+$ satisfy $\| z \| \leqq 1$. Then, for every $k \geqq 1$ we have either $\| z^n - z^{n+k} \| = 2$ for all $n \geqq 0$ or $\| z^n - z^{n+k} \| \to 0$ as $n \to \infty$. Moreover, $\| z^n - z^{n+k} \| \to 0$ if and only if every $\lambda \in \sigma_r(z) \cap \{\gamma : |\gamma| = 1\}$ satisfies $\lambda^k = 1$.*

3.5. COROLLARY. *Let $z \in A^+$ satisfy $\| z \| \leqq 1$. Then the following are equivalent.*

(i) $\| z^n - z^{n+k} \| = 2$ *for all* $n \geqq 0$ *and for all* $k \geqq 1$.

(ii) $\sigma_r(z) = \{\lambda : |\lambda| = 1\}$.

(iii) $\| \Sigma_{j=1}^p \varepsilon_j z^{n+k_j} \| = p$ *for all* $n \geqq 0$ *and for every choice of integers* $1 \leqq k_1 < \cdots < k_p$ $(\varepsilon_j = \pm 1)$.

PROOF. (iii)⇒(i) is immediate and (i)⇒(ii) follows from the preceding results.

(ii)⇒(iii): Let λ, $|\lambda| = 1$ be such that the numbers λ^{n+k_j} are distinct for $j = 1, 2, \ldots, p$. The result then follows from 3.1. ■

We say that $\sigma_r(z)$ has a *non-trivial finite group structure* if it is a finite union of non-trivial groups of roots of unity.

3.6. COROLLARY. *Let $z \in A^+$, $\| z \| \leqq 1$. Then $\sigma_r(z)$ has a non-trivial finite* *group structure if and only if there exists a unique integer* $k_0 \geqq 2$ *such that* $\| z^n - z^{n+k_0} \| \to 0$ *as* $n \to \infty$ *and* $\| z^n - z^{n+k} \| = 2$ *for all* $1 \leqq k < k_0$ *and for all* $n \geqq 0$. *Moreover,* k_0 *is the least common multiple of the orders of the finite groups of which* $\sigma_r(z)$ *is the union.*

PROOF. If $\sigma_r(z)$ has a non-trivial group structure, then $\| z^n - z^{n+1} \| \not\to 0$ by the result of Katznelson and Tzafriri. Also, $\| z^n - z^{n+k} \| \neq 2$ for all $k \geqq 1$ and for all $n \geqq 0$ by 3.5. It follows that there exists an integer as claimed.

On the other hand, if such an integer k_0 exists, $\| z^n - z^{n+1} \| \not\to 0$ since $k_0 \geqq 2$, and also $\| z^n - z^{n+k} \| \neq 2$ for all n and k. Therefore, $\sigma_r(z)$ has a non-trivial finite group structure.

By 3.4, $\lambda^{k_0} = 1$ for every $\lambda \in \sigma_r(z)$ and so k_0 is a common multiple of the orders of the subgroups of roots of unity in $\sigma_r(z)$. But, if $k < k_0$, there exists $\lambda \in \sigma_r(z)$ such that $\lambda^k \neq 1$. This shows that k_0 is the least common multiple. ■

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